MATHEMATICS MADE EEEZY

By

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Combinatorics

FACTORIAL

Combinatorial Analysis, which consists of factorial, Permutation and combination wasdiscovered by **Christian Kramp of France**, **Thomas Blaise Pascal of France** and **Pierre de Fermat of France**.

It was first introduced into text in the city of Strasbourg by scientist Christian Kramp in the year 1808.

To be able to comprehend the topic 'Combinatorial and Permutational Analysis', the concept of factorial must be fully understood.

FACTORIAL ANALYSIS.

Factorial

can be defined as the reduction of an item or number from its original value to unity, i.e 1.

Example

1. $n! = n(n - 1)(n - 2)(n - 3)............(n - n + 1)$
2. $4! = 4(4 - 1)(4 - 2)............(4 - 4 + 1)$
3. $20! = 20 \times 19 \times 18 \times 17 \times 16 \times ......\ 1.$

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Special Cases

4. $1! = 1$
5. $0! = 1$

That is,

$$0! = 1! = 1.$$

1. Simplification of Factorials.

This involves the reduction of factorials into values through the factorial principle.

Example1.

Simplify

(a). 4! b). $(3!)^2$ (c). $(3!)^2 \times (2!)^2$ ((d). $^{10!}/_{8!}$

Solution

(a). $4! = 4 \times 3 \times 2 \times 1$
$= 24$

(b). $(3!)^2 = 3! \times 3!$
$= (3 \times 2 \times 1)(3 \times 2 \times 1)$
$= 6 \times 6$
$= 36$

(c). $(3!)^2 \times (2!)^2 = 3! \times 3! \times 2! \times 2!$
$= (3 \times 2 \times 1)(3 \times 2 \times 1)(2 \times 1)(2 \times 1)$
$= 6 \times 6 \times 2 \times 2$
$= 144$

(d). $\frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!}$
$= 10 \times 9$
$= 90$

2. Conversion of product of items into factorials.

To convert products of factors into factorials, use

$$\frac{\textbf{(largest factor)!}}{\textbf{(smallest - 1)!}}$$

e.g Convert a(a - 1)(a - 2)(a - 3)(a - 4) into factorial

$$a(a - 1)(a - 2)(a - 3)(a - 4) = \frac{a!}{(4 - 1)!}$$

Example 2.

Convert the following into factorials

(a). $8 \times 7 \times 6 \times 5$ (b). $10 \times 9 \times 8 \times 7$

(c). $n(n - 1)(n - 2)(n - 3)$ (d). $2n(2n - 1)(2n - 2)(2n - 3)$

Solution

(a).

$$8 \times 7 \times 6 \times 5 = \frac{8!}{(5-1)!} = \frac{8!}{4!}$$

(b).

$$10 \times 9 \times 8 \times 7 = \frac{10!}{(7-1)!} = \frac{10!}{6!}$$

(c).

$$n(n-1)(n-2)(n-3) = \frac{n!}{(n-3-1)!} = \frac{n!}{(n-4)!}$$

(d).

$$2n(2n-1)(2n-2)(2n-3) = \frac{2n!}{(2n-4)!}$$

3. Factorisation of factorials.

In the factorisation of factoials, the bigger factorial is first reduced so as to have a smaller and common factorial with the other factorials in the given expression. The expression is then factorised.

The following should be noted when factorising factorials:

1. $(a + b)! \neq a! + b!$ and
   $a! + b! \neq (a + b)!$
   e.g
   $(2 + 3)! \neq 2! + 3!$
   $(5)! \neq 2! + 3!$

2. $(a - b)! \neq a! - b!$ and
$a! - b! \neq (a - b)!$
e.g
$(5 - 2)! \neq 5! - 2!$
$3! \neq 5! - 2!$

3. $(a \times b)! \neq a! \times b!$ and
$a! \times b! \neq (a \times b)!$
e.g
$(2 \times 3)! \neq 2! \times 3!$
$6! \neq 2! \times 3!$

4. $(a / b)! \neq a! / b!$ and
$(a / b)! \neq a! / b!$
e.g
$(4 / 2)! \neq 4! / 2!$
$2! \neq 24 / 2$

Example 3

Factorise the following

(a). 14! + 13!

(b). 14! - 10(13!)

(c). (n + 3)! + (n + 2)!

Solution

(a). 14! + 13!
= (14 x 13!) + 13!
= 13!(14 + 1)
= 13!(15).

(b). $14! - 10(13!)$

$= (14 \times 13!) - 10(13!)$

$= 13!(14 - 10)$

$= 13!(4)$

(c). $(n + 3)! + (n + 2)!$

$= \{(n + 3)(n + 3 - 1)!\} + (n + 2)!$

$= \{(n + 3)(n + 2)!\} + (n + 2)!$

$= (n + 2)!\{(n + 3) + 1\}$

$= (n + 4)(n + 2)!$

The following should be noted when factorising factorials:

1. $(a + b)! \neq a! + b!$ and

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   e.g

   $(2 + 3)! \neq 2! + 3!$

   $(5)! \neq 2! + 3!$

2. $(a - b)! \neq a! - b!$ and

   $a! - b! \neq (a - b)!$

   e.g

   $(5 - 2)! \neq 5! - 2!$

   $3! \neq 5! - 2!$

3. $(a \times b)! \neq a! \times b!$ and

   $a! \times b! \neq (a \times b)!$

   e.g

   $(2 \times 3)! \neq 2! \times 3!$

   $6! \neq 2! \times 3!$

4. $(a / b)! \neq a! / b!$ and
$(a / b)! \neq a! / b!$
e.g
$(4 / 2)! \neq 4! / 2!$
$2! \neq 24 / 2$

EXAMPLE 3
Factorise the following
(a). $14! + 13!$
(b). $14! - 10(13!)$
(c). $(n + 3)! + (n + 2)!$

Solution

(a). $14! + 13!$
$= (14 \times 13!) + 13!$
$= 13!(14 + 1)$
$= 13!(15).$

(b). $14! - 10(13!)$
$= (14 \times 13!) - 10(13!)$
$= 13!(14 - 10)$
$= 13!(4)$

(c). $(n + 3)! + (n + 2)!$
$= \{(n + 3)(n + 3 - 1)!\} + (n + 2)!$
$= \{(n + 3)(n + 2)!\} + (n + 2)!$
$= (n + 2)!\{(n + 3) + 1\}$
$= (n + 4)(n + 2)!$

PERMUTATION

This involves the ‘possible ordered arrangement’ of elements or items.
The arrangement relates to either
i. sitting
ii. linning or
iii. standing
of items, objects or elements.
Permutation can be identified by the inclussion of ‘digit number’ in the given question.
Permutation is denoted by

$^{n}P_{r}$ or **P[n,r]** where $n \geq r$.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Example 4.
Find the value of the following
i. $^{5}P_{3}$ ii. $^{10}P_{2}$ iii. $^{6}P_{6}$

Solution

i. $^{5}P_{3} = \frac{5!}{(5-3)!}$

$= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$

$= 5 \times 4 \times 3$

$= 60.$

ii. $^{10}P_{2} = \frac{10!}{(10-2)!}$

$= \frac{10 \times 9 \times 8!}{8!}$

$= 90.$

iii. $^{6}P_{6} = \frac{6!}{(6-6)!}$

$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{0!}$

recall that 0! = 1

= 720

Types of permutation

1. Permutation of unlike or different items
2. Permutation of like or identical items
3. Conditional Permutation
4. Cyclic Permutation.

1. Permutation of different or unlike items

This is an arrangement of items or objects with unque appearance. That is, no item is repeated. Such permutation is obtained using

No. of ways

= (total no. of items)!

= n!

Where n is the number of items given.

Example 5.

(a). In how many ways can the letters of the word 'LAGOS' be arranged?

Solution

In the word 'LAGOS', no single item (letter) is repeated and there are five items (letters) involved. Hence,

n = 5, using

no. of ways = n!

n! = 5!

= 5 x 4 x 3 x 2 x 1

= 120 ways.

(b). In how many ways can the letters of the word 'BELT' be arranged ?

Solution

n = 4
n! = 4!ways
= 4 x 3 x 2 x 1
= 24 ways.

2. Permutation of like or Identical items.

This is an arrangement of items or objects which are repeated. That is, items that are identical. Such permutation is obtained using

No. of ways =

$$\frac{\textbf{(total no. of items)!}}{\textbf{(no. of repeated items)!}}$$

No. of ways =

$$\frac{\textbf{n!}}{\textbf{(no. of repeated items)!}}$$

Example 6.

(a). In how many ways can the letters of the word 'BEGINNING' be arranged ?.

Solution

BEGINNING

total no. of items = 9

Rep. items: G = 2, N = 3, I = 2

$$\text{No. of ways} = \frac{\text{(total no.of items)!}}{\text{(total no.of items)!}}$$

$$= \frac{9!}{2! \text{ x } 3! \text{ x } 2!}$$

$$= \frac{9 \text{ x } 8 \text{ x } 7 \text{ x } 6 \text{ x } 5 \text{ x } 4 \text{ x } 3!}{2! \text{ x } 2! \text{ x } 3!}$$

= 15120 ways.

(b). In how many ways can the letters of the word 'ELEBOLO' be arranged ?

Solution

ELEBOLO

total no. of items = 7

Rep. items: E = 2, L = 2, O = 2

$$\text{No. of ways} = \frac{(\text{total no.of items})!}{(\text{no.of repeated items})!}$$

$$= \frac{7!}{2! \times 2! \times 2!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2! \times 2! \times 2!}$$

$$= 210.$$

3. Conditional Permutation.

This is an arrangement of items or objects in a specified order.That is, a certain condition is provided for the arrangement. In other words, the question will specify which two items must 'be together' or must 'not be together'.

<u>'be together'</u>

the two objects that must be together are regarded as a digit (1) and added to the remaining number of digits and the factorial is taken. Such permutation is obtained using

No. of ways = (Remainder + 1)!

No. of ways = (R + 1)!

'not be together'

In this type of permutation, two specified objects must not be together.**'not be together'**

Such permutation is obtained using

No of ways =

n! - (R + 1)!

(no. of Rep. item)!

where n is the total no. of items, R the remainder and Rep. the repeated items.

Example 7.

The letters of the word '**SHAPPEY**' are to be arranged. In how many of these arrangements:

(a). do the P's come together

(b). do the P's stay apart.

Solution

(a). P's come together

SHAPPEY

no.of items (n) = 7

no of repeated tems i.e P = 2

note: the two Ps are subtracted from total no of items to get the remainder

Remainder = 7 - 2 = 5

using

No. of way

= (Remainder + 1)!

= (5 + 1)!

= 6!

= 720 ways.

(b). do the P's stay apart.
using
No of ways =

$$\frac{n!}{(\text{no. of rep. items})} - (R + 1)!$$

$$= \frac{7!}{2!} - 6!$$

= 2520 - 720
= 1800 ways.

Example 8

Five people are to be arranged in a row for a group photograph. How many arrangements are there if a married couple in the group insist on sitting next to each other ?

Solution

no.of people (n) = 5 two (couple) must sit next to each other
note: the two are subtracted from total number of people to get the remainder

Remainder = 5 - 2 = 3, R = 3

using

No. of ways = (Rem. + 1)!
= (3 + 1)!
= 4!
= 24 ways.

4. Cyclic Permutation.

This permutation deals with the arrangement of items or objects in a circular pattern. This is done by placing any one of the items in a fixed position (at the centre) and the other items are then arranged in a circular path around it. The number of ways is obtained using

No. of ways = (n - 1)!

where n is the number of items.

NB: This type of arrangement is also called **RING PERMUTATION.**

Example 9.

In how many ways can the letters A, B, C, D and E can be arranged on a circle.

Solution

There are five letters, A, B, C, D and E, i.e n = 5
any one of the five must be at the centre of the circle, thus leaving four letters. Any of these four letters can occupy any four positions on the circle,
which gives

$4! = 4 \times 3 \times 2 \times 1$
$= 24$ ways

Example 10

In how many ways can ways six beauty pagents be arranged round a circular table to await the result.

Solution

n = 6

No. of ways = $(n - 1)!$
$= (6 - 1)!$
$= 5!$
$= 120$ ways.

COMBINATION

This is a branch of statistics which deals with the selection of objects from a given number of objects using the factorial principles. It is denoted by either **^nC_r, or [n,r]**

^nC_r means ' selection of r objects from n objects' **or** 'choosing of r objects from n objects". It is obtained using

$$^nC_r = \frac{n!}{(n-r)!r!}$$

It is also helpful in solving problems under probability.

Example 11

Find the value of the following

i. $^7C_3$ ii. $^7C_3 + {}^5C_3$

iii. $^nC_3$

Solution

i. $^7C_3 = \frac{7!}{4! \times 3!}$

$= \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!}$

$= 35$

ii. $^7C_3 + {}^5C_3$

$^7C_3 = \frac{7!}{4! \times 3!}$

$= \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!}$

$= 7 \times 5$

$= 35$

$$^5C_3 = \frac{5!}{2! \times 3!}$$

$$= \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!}$$

$$= 10$$

$$\therefore C_3 + {}^5C_3 = 35 + 10$$

$$= 45.$$

iii. $$^nC_3 = \frac{n!}{(n-3)! \times 3!}$$

$$= \frac{n(n-1)(n-2)(n-3)!}{(n-3)! \times 3!}$$

$$= \frac{n(n-1)(n-2)}{6}$$

Combination and Permutation can easily be identified in a question with the help of this secret keys

*** formation of digit numbers (Permutation)**

*** formation of groups or committee (Combination)**

Example 12.

In how many ways can 4 objects be selected from a set of 7 objects.

Solution

no. of given objects (n) = 7

no. of selected objs. (r) = 4

using $$^nC_r = \frac{n!}{(n-r)!r!}$$

$$^7C_4 = \frac{7!}{(7-4)! \times 4!}$$

$$= \frac{7!}{4! \times 3!}$$

$$= \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!}$$

$$= 35$$

EQUATIONS INVOLVING COMBINATION AND PERMUTATION.

These are mathematical statements involving combination and permutation connected with the equallity sign and in which a variable is to be determined using the the factorial principles.

It should however be noted that the value of the variable to be determined **can not be negative and also a fraction. That is, it must be a positive non- zero integer.**

Example 13.

If $^{n}C_{r} = \frac{n!}{(n - r)!r!}$ then

$^{n+1}C_{n-1}$ is

Solution

$^{n+1}C_{n-1}$

$$= \frac{(n + 1)!}{\{(n + 1) - (n - 1)\}!(n - 1)!}$$

$$= \frac{n(n + 1)}{2}$$

Example 14.

Find the value of n, if $^{n}C_{2} = 45$.

Solution

$^{n}C_{2} = 45$

using

$$^{n}C_{r} = \frac{n!}{(n - r)!r!}$$

$$^{n}C_{2} = \frac{n!}{(n - 2)!2!} = 45$$

$$\frac{n!}{(n - 2)!2!} = 45$$

$$\frac{n(n - 1)(n - 2)!}{2 \times (n - 2)!} = 45$$

$$\frac{n(n - 1)}{2} = 45$$

cross multiply

$n^2 - n - 90 = 0$, factorising

$n = 10$ or $n = -9$, since $n \neq -9$

$\therefore$ $n = 10$.